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On the uniqueness of the surface sources of evoked potentials

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The uniqueness of a surface density of sources localized inside a spatial region R and producing a given electric potential distribution in its boundary B_0 is revisited. The situation in which R is filled with various subregions, each one having a definite constant value for the electric conductivity is considered. It is argued that the knowledge of the potential in all B_0 fully determines the surface-located sources for a general class of surfaces supporting them and also a wide type of those sources. The class of surfaces can be defined as a union of an arbitrary but finite number of open or closed surfaces. The only restriction upon them is that no one of the closed surfaces contains inside it another (nesting) of the closed or open ones. The types of sources are surface charge densities and double layer (dipolar) densities for the open surfaces and more restrictively, only surface charge densities for the closed ones. A two-dimensional analytically solvable example illustrating the drastic appearance of uniqueness after arbitrarily small holes are opened in nested surfaces is discussed.

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I. INTRODUCTION

The uniqueness problem for the sources of the evoked potential in the brain is a relevant research question due to its role in the development of cerebral electric tomography [1–4]. For a long time, it has been known that the general inverse problem in the determination of volumetric sources from the measurement of the potential at a surface is not solvable in general [5,6]. However, under additional assumptions about the nature of the sources, solutions can be obtained [7–9]. The supplementary assumptions can be classified into two groups: the physically grounded ones, which are fixed by the nature of the physical problem and the ones that are imposed by invoking their mathematical property of determining a solution, but having a weak physical foundation. The resumed situation implies that the determination of physical conditions implying the uniqueness of the sources for the evoked potentials remains an important subject of study. Results in this direction could avoid the imposition of artificial conditions altering the real information on the sources to be measured.

The question to be considered in this work is the uniqueness of the sources for evoked potentials under the supposition that these sources are localized over surfaces. This issue was also treated in Ref. [1] by including some specially defined volumetric sources. The concrete aim here is to present a derivation of the results enunciated in Ref. [1] for the case of open surfaces and to generalize it for a wider set of surfaces including closed ones.

It should be specified that as in Ref. [1], here we will assume that the evoked potential is measured over a closed surface. Clearly, this condition is not satisfied in the practical situations where the field in area of the neck cannot be determined. However, accepting that the sources are known to be localized near the top of the head, for example, the specific conclusions of this paper could be expected to apply approximately. For example, situations for which it could be expected to occur are such ones in which the sources of the EEG are identified to be localized in some open region of the cortex surface. The conditions for the practical applications of the discussion will be deferred for future analysis, since they need a different sort of mathematical treatment.

We consider that the results enunciated in Ref. [1] are valid and useful ones. Even more, we think that a relevant merit of that paper is to call attention to the possibility of the uniqueness of classes of surface density of sources. Specifically, in our view, the conclusion stated there about the uniqueness of the sources of evoked potentials as restricted to sources distributed in open surfaces is effectively valid. In the present work, the central aim is to extend the result for a wider set of surfaces including closed ones by also furnishing an alternative way to derive the uniqueness result. The uniqueness problem for the special class of volumetric sources discussed in Ref. [1] is not considered here in any way.

The physical system under consideration for the proof of the uniqueness is formed by various volumetric regions, each of them having a constant value of the conductivity, separated by surface boundaries at which the continuity equations for the electric current is obeyed. The particular subset of the open surfaces, in a more general way, is also allowed to support continuous double layer (dipolar) surface source distributions, which are frequently used for physical applications in evoked potentials research.

The precise definition of the generators under examination is the following. The sources are assumed to be defined by continuous and smooth surface charge densities lying over an arbitrary but finite number of smooth open or closed surfaces. The opened surfaces, as mentioned above, can also support double layer surface densities. The unique constraint to be imposed on these surfaces is that there is no nesting among them. That is, there is no closed surface at which interior another open or closed of

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the surfaces resides. This class of supports expands the one considered in Ref. [1]. It should be stressed that the boundaries between the interior regions are not restricted by the "non-nesting" condition. That is, the fact that the skull and the few boundaries between cerebral tissues can be visualized as nearly closed surface does not pose any limitation on the conclusion. The "non-nesting" condition should be valid only for the surfaces in which the sources can be expected to reside. For example, if by any means we are sure that the sources stay at the cortex surface, then the uniqueness result apply whenever the portion of the cortex implied does not contain any closed surface.

It should be recognized that the claimed result looks somewhat suspicious after considering that nestedlike surfaces can be drastically transformed in non-nested ones by arbitrarily small holes opened in them. This implies that, so strangely as it appears, the inverse problems for the considered charge distributions should be turned off discontinuously from having nonunique to unique solutions. However, in support of the possible occurrence of this effect is that the discussion is related to a continuous system in which the potential in the outside boundary is assumed to be exactly known over a continuous surface. Then, it comes to mind that the opening of the hole would lead to a small but nonvanishing space-dependent potential at the outside of the now "almost nesting" surface. Since we are assuming the capability of performing an exact measure of the potential, no matter the weakness of the field, it becomes more natural to accept the possibility for the potential to determine the sources uniquely. We interpret this state of affairs in the following sense: for the cases in which the holes are really small, to determine the sources in a practical discretization of the considered problem will be very much difficult than for the cases in which the holes are bigger. In the former case the sensitivity of the measuring device needed for determining uniquely the sources should be much greater in dependence of the smallness of the holes. That is, in spite of the correctness of the claimed result, the difficulty to experimentally check the uniqueness in practical experiences will depend on the concrete situations.

In order to show the realization of the above general argument within a specific example we discuss the two-dimensional (2D) electrostatic solution associated with a line of charges that is parallel to a slotted and also planar grounded conductor. However small but nonvanishing the slot is, it is shown that the measuring of the potential within a whole plane separated from the charges by the slotted conductor, uniquely determines the planar sources. Taking into account that for a rigorously null size of the slot, the inverse problem has no solution, the discontinuous change in the uniqueness character of the problem induced by small drilled holes becomes illustrated.

The paper is organized as follows. An auxiliary property is derived in the form of a theorem in Sec. II. In Sec. III the proof of uniqueness for the kind of sources defined above is presented. Finally, a two-dimensional illustration corresponding to a slotted plane and a line of charges parallel to it, is discussed in Sec. IV.

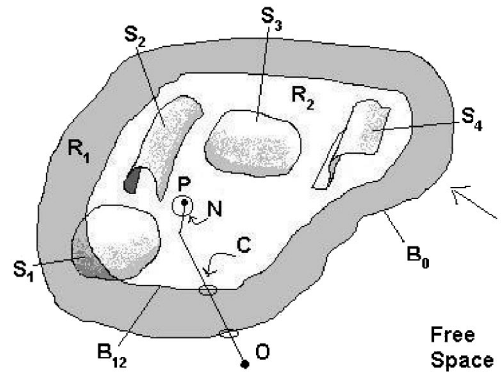

FIG. 1. An illustration of a simply connected region R constituted in this case by only two simply connected subregions R_1 and R_2 having a boundary B_{12}. The boundary with free space is denoted by B_0. The set of non-nesting surfaces S have four elements S_i, $i=1,\ldots,4$ two of them open and other two closed ones. A piecewise straight curve C joining any interior point P of R and a point O in the free space is also shown.

II. GREEN THEOREM AND FIELD VANISHING CONDITIONS

Let us consider the potential ϕ generated by a source distribution concentrated in the "non-nested" set of open or closed surfaces defined in Sec. I, which at the same time are contained within a compact and simply connected spatial region R. The set R, as explained before, is formed by various connected subregions R_i, $i=0,1,\ldots,n$ each of them filled with a substance having a constant conductivity σ_i. Also, let B_{ij} be the possibly but unnecessarily existing boundary between the subregions R_i and R_j, and B_0 be the boundary of R. For the sake of a physical picture, we can interpret B_0 as the surface of the skull, R as the interior of the head and the subregions R_i as the ones containing the various tissues within the brain. It is defined that the exterior space of the head corresponds to R_0. In addition, let S_i, $i=1,\ldots,m$ be the surfaces pertaining to the arbitrary but finite set S of non-nested open or closed surfaces in which the sources are assumed to be localized. The above-mentioned definitions are illustrated in Fig. 1.

Then, the Poisson equation satisfied by the potential ϕ in the interior points of R (but which are outside the boundaries between the internal regions R_i) can be written as

$$\vec{\nabla}\cdot[\sigma(\vec{x})\vec{\nabla}\phi(\vec{x})]=g(\vec{x}), \tag{1}$$

$$g(\vec{x})=-\vec{\nabla}\cdot\vec{J}(\vec{x}), \tag{2}$$

where $\vec{J}$ are the impressed currents (for example, generated by the neuron firings within the brain) and the space dependent conductivity is defined by

$$\sigma(\vec{x})=\sigma_i \quad \text{for} \quad \vec{x}\in R_i . \tag{3}$$

Note, that for the points $\vec{x}$ in which the above Poisson equation is defined, the conductivity $\sigma(\vec{x})$ is effectively a constant and can be extracted from the divergence operator.

It should be noticed that the conductivities are different from zero only for the internal regions of R. The vacuum outside is assumed to have zero conductivity and the field satisfying the Laplace equation. In addition, outside the support of the sources where $g=0$ the Laplace equation is also satisfied except at the separations between the regions R_i, where the usual boundary conditions (within the static approximation) associated with the continuity of the electric current flowing through these boundaries, takes the form

$$\sigma_i \frac{\partial \phi}{\partial n_i}\bigg|_{x\in B_{ij}} = -\sigma_j \frac{\partial \phi}{\partial n_j}\bigg|_{x\in B_{ij}}, \qquad (4)$$

where ∂n_i symbolizes the directional derivative along a line normal to B_{ij} but taken in the limit of $x \rightarrow B_{ij}$ from the side of the region R_i. The direction defining the derivative is taken as the one going from R_i to R_j, which justifies the minus sign in Eq. (4).

A main property is employed in this work for obtaining the claimed result. In the form of a theorem for a more precise statement, it is expressed as follows.

Theorem 1. Let us consider a differentiable surface S^* that divides an open ball R^* into two connected and open subregions R_+^* and R_-^* and assume that the field φ satisfies the Laplace equation within, say, R_+^*. Consider also a smaller ball Q with its center lying on S^*, and its spherical boundary having a minimal nonvanishing distance to the boundary of R^*. Define Q_+ as equal to the intersection of Q with R_+^*. Then, if the gradient of φ tends to vanish on approaching any point of the intersection of S^* with Q while coming from the interior of Q_+, it follows that φ is constant over any open set contained within R_+^*.

As a first stage in the derivation of this property, let us consider the Green Theorem as applied to an interior region Q_+^ϵ of the open set Q_+ defined in the Theorem 1, inside which the field φ satisfies the Laplace equation. Then, the theorem expresses φ evaluated at a particular interior point $\vec{x}$ in terms of itself and its derivatives at the boundary $B_{Q_+^\epsilon}$ as

$$\varphi(\vec{x}) = \int_{B_{Q^\epsilon +}} \frac{d\vec{s}\,'}{4\pi} \cdot \left[\frac{1}{|\vec{x}-\vec{x}\,'|} \vec{\nabla}_{x'} \varphi(\vec{x}\,') - \vec{\nabla}_{x'} \left(\frac{1}{|\vec{x}-\vec{x}\,'|} \right) \varphi(\vec{x}\,') \right], \qquad (5)$$

where the integral is running over the boundary surface $B_{Q^\epsilon +}$ that is spanned by the coordinates $\vec{x}\,'$. This relation expresses the potential as a sum of surface integrals of the continuous and bounded values of φ and its derivatives. Now, it is possible to set the boundary of Q_+^ϵ to approach the corresponding boundary of Q_+ in the limit $\epsilon \rightarrow 0$ in such a way that no point of S^* is touched in the limiting process. It can be remarked that this consideration for the limiting process is necessary for the cases in which S^* coincides with a boundary between different conductivity regions or a surface supporting the sources. Exactly at the surface the gradient or the field itself could be undefined.

After performing the limit process and due to the assumption of the vanishing of the gradient over S^* the first integral appearing in Eq. (5) will reduce its domain to the spherical part B_{Q+sphe} of the boundary of Q_+. Moreover, since the integral of the gradient over any curve inside Q^+ that tends to be parallel but nontouching S^*, should approach zero because of the vanishing condition for the gradient, the limiting value of the potential for any point tending to S^* from the inside of Q^+, becomes a certain constant value φ_0. This property allows us to write for Eq. (5) in the limit $\epsilon \rightarrow 0$, the expression

$$\begin{aligned} \varphi(\vec{x}) &= \varphi_0 + \lim_{\epsilon -> 0} \int_{B_{Q_+^\epsilon}} \frac{d\vec{s}\,'}{4\pi} \cdot \left[\frac{1}{|\vec{x}-\vec{x}\,'|} \vec{\nabla}_{x'} \varphi(\vec{x}\,') - \vec{\nabla}_{x'} \left(\frac{1}{|\vec{x}-\vec{x}\,'|} \right) [\varphi(\vec{x}\,') - \varphi_0] \right] \\ &= \varphi_0 + \int_{B_{Q+sphe}} \frac{d\vec{s}\,'}{4\pi} \cdot \left[\frac{1}{|\vec{x}-\vec{x}\,'|} \vec{\nabla}_{x'} \varphi(\vec{x}\,') - \vec{\nabla}_{x'} \left(\frac{1}{|\vec{x}-\vec{x}\,'|} \right) [\varphi(\vec{x}\,') - \varphi_0) \right], \end{aligned} \qquad (6)$$

where the following identity has been employed:

$$\varphi_0 = -\int_{B_{Q_+^\epsilon}} \frac{d\vec{s}\,'}{4\pi} \cdot \vec{\nabla}_{x'} \left(\frac{1}{|\vec{x}-\vec{x}\,'|} \right) \varphi_0, \quad \vec{x} \in Q_+^\epsilon.$$

The last integral in Eq. (6) has a domain that contains points having a finite distance from the center of Q. Therefore, it implies that the potential φ is an analytical function of all the components of a point $\vec{x}$ lying inside Q_+ with the only condition for it to have a finite minimal distance from the points in B_{Q+sphe}. An important property of the expression (6) for φ is that it should be also valid for the points arbitrarily near but not pertaining S^*, independently of the possible boundary character of this surface or not. It should be noticed that although Eq. (6) is not valid for the description of the real problem at the points of S^*, it is yet a well-defined analytical function of the coordinates at all the points of S^* having a finite distance from B_{Q+sphe}.

Further, let us consider an arbitrary point $P \in S^*$ being inside the region Q, but having a finite distance from the spherical part of the boundary of Q_+, that is B_{Q+sphe}. The condition for S^* to be differentiable in a sufficiently small neighborhood $N_P \subset S^*$ can be now expressed by specifying the coordinates of this surface as follows:

$$x_1(s_1,s_2,0) \equiv s_1,$$

$$x_2(s_1,s_2,0) \equiv s_2, \qquad (7)$$

$$x_3(s_1,s_2,0)\equiv g(s_1,s_2),$$

where, clearly the parameters defining the surface are the two specific coordinates x_1 and x_2; and $g(s_1,s_2)$ is a differentiable function of both variables. The coordinates are measured from an origin situated at the point P. Now, and within a sufficiently small but finite neighborhood of P, it is possible to define a curvilinear coordinate system in the following way:

$$x_1(s_1,s_2,s_3)\equiv s_1+s_3\,n_1(s_1,s_2),$$

$$x_2(s_1,s_2,s_3)\equiv s_2+s_3\,n_2(s_1,s_2), \tag{8}$$

$$x_3(s_1,s_2,s_3)\equiv g(s_1,s_2)+s_3\;\;n_3(s_1,s_2),$$

where $\vec{n}\equiv(n_1(s_1,s_2),n_2(s_1,s_2),n_3(s_1,s_2))$ is a normal unit vector to S^* at the surface point $\vec{x}(s_1,s_2,0)$ of S^* and s_3 is the distance from the general coordinate point $\vec{x}(s_1,s_2,s_3)$ to a corresponding surface point $\vec{x}(s_1,s_2,0)$. Therefore, by construction $\vec{x}(s_1,s_2,s_3)$ is assumed to be along a line normal to S^* at the point $\vec{x}(s_1,s_2,0)$.

Taking the derivatives of the coordinates (8) follows that

$$\sum_{i=1}^{3}\frac{\partial x_i(s)}{\partial s_3}\frac{\partial x_i(s)}{\partial s_1}=\sum_{i=1}^{3} n_i(s_1,s_2)\left(\frac{\partial x_i}{\partial s_1}(s_1,s_2,0)+s_3\frac{\partial}{\partial s_1}n_i(s_1,s_2)\right)=0,$$

$$\sum_{i=1}^{3}\frac{\partial x_i(s)}{\partial s_3}\frac{\partial x_i(s)}{\partial s_2}=\sum_{i=1}^{3} n_i(s_1,s_2)\left(\frac{\partial x_i}{\partial s_2}(s_1,s_2,0)+s_3\frac{\partial}{\partial s_2}n_i(s_1,s_2)\right)=0, \tag{9}$$

$$s=(s_1,s_2,s_3),\quad x=(x_1,x_2,x_3).$$

These relations follow because the derivative of a unit vector is always normal to it and also because the vectors $\partial\vec{x}(s_1,s_2)/\partial s_{1,2}$ are tangent to the surface and therefore orthogonal to $\vec{n}(s_1,s_2)$. In addition, the identity

$$\sum_{k=1}^{3}\frac{\partial x_i(s)}{\partial s_k}\frac{\partial s_k(x)}{\partial x_j}=\delta_{ij}$$

expresses that the matrix $\partial s_k(x)/\partial x_j$ is the inverse of $\partial x_i(s)/\partial s_k$. Now, it could be noticed that from Eq. (8) it follows that $\partial x_i(s)/\partial s_k$ is an analytic function of s_3. The same property is not happening with respect to the other variables s_1,s_2, because the surface S^* is only supposed to be differentiable. Therefore, it follows that the components of the inverse matrix $\partial s_k(x)/\partial x_j$ and its derivatives, which are needed below, also will be analytic functions of s_3. Also, it will be useful to notice here that Eq. (6) implies that the field φ is also an analytic function of s_3 for fixed s_1,s_2.

Now, the Laplace equation for φ in terms of the s coordinates can be written as

$$\frac{\partial^2}{\partial s_3\partial s_3}\varphi(s)+\frac{1}{h(s)}\sum_{i=1}^{3}\frac{\partial^2 s_3}{\partial x_i\partial x_i}\frac{\partial}{\partial s_3}\varphi(s)+\frac{1}{h(s)}\sum_{i=1}^{3}\sum_{\sigma=1}^{2}\frac{\partial^2 s_\sigma}{\partial x_i\partial x_i}\frac{\partial}{\partial s_\sigma}\varphi(s)+\frac{1}{h(s)}\sum_{i=1}^{3}\sum_{\sigma=1}^{2}\sum_{\sigma'=1}^{2}\frac{\partial s_\sigma}{\partial x_i}\frac{\partial s_{\sigma'}}{\partial x_i}\frac{\partial^2}{\partial s_\sigma\partial s_{\sigma'}}\varphi(s)=0, \tag{10}$$

$$h(s)=\sum_{i=1}^{3}\frac{\partial s_3}{\partial x_i}\frac{\partial s_3}{\partial x_i}. \tag{11}$$

The absence of the terms containing mixed derivatives over s_3 and $s_{1,2}$ is a consequence of Eq. (9). From the analyticity with respect to s_3 of the Jacobian $\partial s_i/\partial xj$ and its derivatives over the s variables it follows that all the coefficients in the Laplace equation (10) are also analytic functions of the s_3 variable. Furthermore, since the vector $\partial s_3/\partial x_i$ is one of the rows of the Jacobian, its square modulus $h(s)$ cannot vanish at the point P, thus allowing the division by it in some sufficiently small but finite neighborhood.

Next, the above-mentioned analyticity of φ with respect to the variable s_3 permits to write the expansion

$$\varphi(s)=\sum_{n=0}^{\infty} f_n(s_1,s_2)s_3^n, \tag{12}$$

where the coefficients f_n are functions of s_1,s_2. Thus, let us substitute Eq. (12) in Eq. (10) and take into account that all the coefficient functions in Eq. (10) are analytically depending on s_3, a property that allows to multiply their series expansions. After that, the necessary vanishing of the coefficients of the Taylor series to which Eq. (10) reduces, implies the following kind of recurrence relations for the functions f_n:

$$f_{n+2}(s_1,s_2)=\frac{1}{(n+1)(n+2)}\times\sum_{m=0}^{n}\Bigg[(m+1)c_{n-m}(s_1,s_2)f_{m+1}(s_1,s_2)+\sum_{\sigma=1}^{2}c_{n-m}^{\sigma}(s_1,s_2)\frac{\partial}{\partial s_\sigma}f_m(s_1,s_2)$$

$$+\sum_{\sigma=1}^{2}\sum_{\sigma'=1}^{2} c_{n-m}^{\sigma,\sigma'}(s_1,s_2)\frac{\partial}{\partial s_\sigma \partial s_{\sigma'}} f_m(s_1,s_2)\Bigg],$$

$$n=0,1,\ldots\infty. \tag{13}$$

Relations (13) imply that the coefficients f_n are determined by their values for lesser values of the index n. Let us study below what are the conditions imposed on the f_n by the vanishing of the gradient at the points of S^*. First, the gradient can be expressed as

$$\frac{\partial\varphi}{\partial x_i}=\sum_{j=1}^{3}\frac{\partial s_j}{\partial x_i}\frac{\partial\varphi}{\partial s_j}=\sum_{j=1}^{3}\frac{\partial s_j}{\partial x_i}\sum_{n=0}^{\infty}\left(\frac{\partial f_n(s_1,s_2)}{\partial s_j}+(n+1)f_{n+1}(s_1,s_2)\,\delta_{j3}\right)s_3^n. \tag{14}$$

Since these derivatives all vanish at all the points of S^*, relation (14) in the limit $s_3\rightarrow 0$ leads to

$$\frac{\partial f_0(s_1,s_2)}{\partial s_1}=0,$$

$$\frac{\partial f_0(s_1,s_2)}{\partial s_2}=0, \tag{15}$$

$$f_1(s_1,s_2)=0.$$

But, thanks to the recurrence character of Eq. (13) and the conditions (15) for the first two coefficients f_n, it follows that all these quantities vanish for $n\geqslant 1$. At the same time Eq. (15) implies

$$f_0(s_1,s_2)=f_0=\text{const.}$$

Therefore, it follows that the field φ determined by Eq. (6) is constant within a finite neighborhood of the point P. It should be recalled that in accordance with the conditions of Theorem 1, Eq. (6) was only valid for the internal points of Q_+ (which are not included in S^*). Therefore, the constancy of φ has been shown here only for certain open neighborhood included in Q_+ having no common points with S^*. Physically, this circumstance is expressing the possibility that the surface S^* could be a part of a boundary between regions of different conductivity or a support of the sources.

To finish the proof of the theorem, it rests to show that if gradient of φ vanish within a certain open neighborhood N, included in an arbitrary connected open set O pertaining to the region R_+^* defined in Theorem 1, in which the Laplace equation is obeyed, then the electric field vanish in all O. Consider first that Q is an open set such that $O\subset Q$ and also suppose that the smallest distance from a point in O to the boundary B_Q of Q has the finite value δ. Then, the Green theorem (5) as applied to the region Q expresses that the minimal radius of convergence of φ and its gradient considered as analytical functions of any of the coordinates (measured from the considered point) is equal or greater than δ.

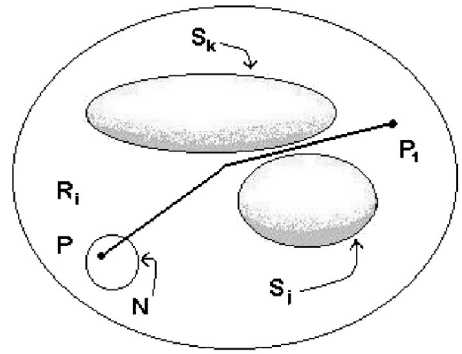

FIG. 2. Picture of the region R_i and the open neighborhood N in which the field φ vanishes exactly. A piecewise straight line curve C joining a point $P\in N$ and certain point P_1 in R_i is also shown.

Imagine now a curve C starting in an interior point P of N and ending at any point P_1 of O. Assume that C is formed by straight lines pieces (see Fig. 2) and that all its points are included in O. It is then possible to define φ as a function of the length of arc s of C as measured from the point P. It should be also valid that in any open segment of C, not including the intersection points of the straight lines pieces, the potential φ and all the components of its gradient $\vec{\nabla}\varphi$, are analytic function of s. Furthermore, let us consider C as partitioned in a finite number of segments of length $\sigma<\delta$. Suppose also, that the intersection points of the straight lines pieces are the borders of some of the segments. It can be noticed that $\vec{\nabla}\varphi$ vanishes in any segment of C starting within N, because it vanishes in N exactly. Furthermore, let us consider also a small open ball $B\subset N$ centered in P and the set of all the curves obtained by parallel translations of the just defined C in the vector joining P with any of the points of B. The set of all the ending points of such curves will define a similar to B open ball with P_1 at its center. It is clear that for a sufficiently small B all such curves will be included in O, since all the points of C pertain to O that is open. Therefore, it follows that in analogy to what happened for the curve C, the components of $\vec{\nabla}\varphi$ vanish at all the segments of the translated curves starting in N. This fact implies that $\vec{\nabla}\varphi$ vanishes in a whole open ball around the ending point of the first segment of the curve C. Since the series expansions of the components of $\vec{\nabla}\varphi$ with respect to the arc length (measured from the intersection point being considered) has by construction a radius of convergence greater than the length of any of the segments, the vanishing $\vec{\nabla}\varphi$ in a whole open ball around one of the intersection points directly implies the vanishing of these quantities within an open ball around the next intersection point. The same argument can be repeatedly done in an iterative way to conclude that $\vec{\nabla}\varphi$ vanishes at the arbitrary ending point P_1 of the curve C. Henceforth, the conclusion of Theorem 1 follows: The potential φ is a constant at any interior point of R_+^*.

III. UNIQUENESS OF THE NON-NESTING SURFACE SOURCES

Let us argue now the uniqueness of the sources which are defined over a set of non-nested surfaces S producing specific values of the evoked potential ϕ at the boundary B_0 of the region R. For this purpose it will be assumed that two different source distributions produce the same evoked potential over B_0. The electrostatic fields in all space associated with those sources should be different as functions defined in all space. They will be called ϕ_1 and ϕ_2. As usual in the treatment of uniqueness problems in the linear Laplace equation, consider the new solution defined by the difference $\phi = \phi_1 - \phi_2$. Clearly φ corresponds to sources given by the difference of the ones associated with ϕ_1 and ϕ_2. It is also evident that ϕ has vanishing values at B_0. Then, since the sources are localized at the interior of R and ϕ satisfies the Laplace equation with zero boundary condition at B_0 and at the infinity, it follows that the field vanishes in all R_0, that is, in the free space outside the head. Therefore, it follows that the potential and the electric field vanish in all B_0 when approaching this boundary from the free space (R_0). The continuity of the potential, the boundary conditions (4) and the irrotational character of the electric field allows to conclude that ϕ and the electric field also vanish at any point of B_0 but now when approaching it from any interior subregion R_i having a boundary B_{i0} with the free space. Moreover, if the boundary surface of any of these regions that are in contact with the boundary of R is assumed to be smooth, then it follows from Theorem 1 that the potential ϕ and the electric field vanish in all the open subsets of R_i, points that are connected through its boundary B_{i0} with free space by curves not touching the surfaces of S. It is clear that this result holds for all the open subsets of these R_i in which Laplace equation is satisfied excluding those that are also residing inside one of the closed surfaces S_i in the set S.

It is useful for the following reasoning to remark that if we have any boundary B_{ij} between two regions R_i and R_j, and the potential ϕ and the electric field vanish in certain open (in the sense of the surface) and smooth regions of it, then Theorem 1 implies that the potential and the electric field also vanish in all the open subsets of R_i and R_j that are outside any of the closed surfaces in S. Since the sources stay at the surfaces in S the field ϕ and its gradient at the internal boundary of the closed surfaces S_i, will not necessarily satisfy the conditions of Theorem 1, even when the field and the gradient can be shown to vanish at the outside.

Let us consider in what follows a point P included in a definite open vicinity of a subregion R_i. Suppose also that P is outside any of the closed surfaces in S. Imagine a curve C that joins P with the free space and does not touch any of the surfaces in S. It is clear that, if appropriately defined, C should intersect a finite number of boundaries B_{ij} including always a certain one B_{j0} with free space. Let us also assume that C is adjusted in a way that in each boundary it crosses, the intersection point is contained in a smooth and open vicinity (in the sense of the surface) of the boundary (see Figs. 1 and 3). Then, it also follows that the curve C can be included in open set O_C having no intersection with the non-

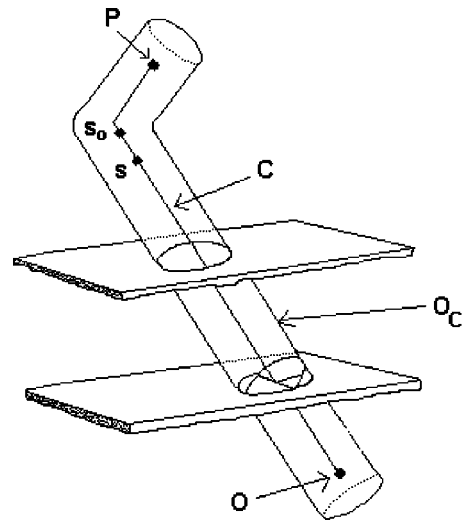

FIG. 3. Scheme of the curve C and the open region O_C containing it.

nested surfaces in S. This is so because the region excluding the interior of the closed surfaces in S is also connected if the S_i are disjoint. But, from Theorem 1 it follows that ϕ and the electric field must vanish in all O_C. This should be the outcome because the successive application of Theorem 1 to the boundaries intersected by the curve C permits to recursively imply the vanishing of ϕ and the electric field in each of the intersections of O_C with the subregions R_i through which C passes. The first step in the recursion can be selected as the intersection of C with B_{j0} at a point that by assumption is contained in an open neighborhood of the boundary B_{j0}. As the electric field and ϕ vanish at free space, the fields in the first of the considered intersection of Oc should vanish. This fact permits to define another open and smooth neighborhood of the next boundary intersected by C in which the field vanish and so on up to the arrival to the intersection with the boundary of the region R_i containing the ending of C at the original point P. Therefore, the electric field and the potential should vanish at an arbitrary point P of R with only two restrictions: (1) P to be contained in an open neighborhood of some R_i and (2) P to reside outside any of the surfaces in S. Thus, it is concluded that the difference solution ϕ and its corresponding electric field, in all the space outside the region containing the sources vanish. Henceforth, it implies that the difference between the two source distributions also should be zero over any of the open surface in the set S. It should be stressed that this conclusion is valid not only for bounded surface charge densities, but also for surface double layer (dipolar) densities. This is necessary because the flux going out from any small piece of the considered surface is zero and moreover, the potential should be strictly constant outside all the closed surfaces, which means that the assumed charge and double layer densities of sources exactly vanish. This completes the proof of the conclusion of Ref. [1] in connection with sources supported by open surfaces. It only remains to be shown that the sources are also null over the closed S_i, whenever, these sources are restricted to be bounded surface densities of charges.

Before continuing with the proof, it is illustrative to exemplify from a physical point of view how the presence of nested surfaces among the S_i destroys the uniqueness. For this aim let us consider that a closed surface S_i has another open or closed surface S_j properly contained inside it. This means that an open set containing S_j is also contained inside S_i. Imagine also that S_i is interpreted as a metal shell connected to the ground; that is, to a zero potential and that the surface S_j is the support of an arbitrary density of sources. As it is known from electrostatics theory, the charge density of a metal connected to the ground is always capable of creating a surface density of charge at S_i such that it exactly cancels the electric field and the potential at the outside of S_i, in spite of the high degree of arbitrariness of the charge densities at the interior. That is, for nested surfaces in S, it is not possible to conclude the uniqueness, because at the interior of a nesting surface, and distributed over the nested ones, arbitrary source distributions can exist that determine exactly the same evoked potential at the outside boundary B_0.

Let us finally show that if no nesting exists the uniqueness also follows when the sources over the closed surfaces are limited to bounded surface densities of charges. Consider any of the closed surfaces, say, S_i. As argued before ϕ and the electric field vanish at any exterior point of S_i pertaining to certain open set containing S_i. Then, the field created by the difference between the sources associated with the two different solutions assumed to exist should be different from zero only at the interior region. That zone, in the most general situation can be filled by a finite number of bodies with different but constant conductivities. The necessary vanishing of the interior field follows from the exact conservation of the lines of forces for the ohmic electric current as expressed in integral form by

$$\int d\vec{s}\cdot\sigma(\vec{x})\vec{E}(\vec{x})=0. \tag{16}$$

Let us consider a surface T defined by all the lines of forces of the current vector passing through an arbitrarily small circumference c that sits on a plane being orthogonal to a particular line of force passing through its center. Let the center be a point at the surface S_i. Because, the above-defined construction, all the flux of the current passing through the piece of surface of S_i (which we will refer to as p) intersected by T is exactly equal to the flux through any intersection of T with another surface determining in conjunction with p a closed region. By selecting a sufficiently small radius for the circumference c it can be noticed that the sign of the electric field component along the unit tangent vector to the central line of forces should be fixed. This is so because otherwise there will be an accumulation of charge in some closed surface. Now, let us consider the fact that the electric field is irrotational and examine a line of force of the current density that must start at the surface S_i. It should end also at S_i, because on another hand the current density will not be divergenceless. After using the irrotational condition for the electric field in the form

$$\oint_C \vec{E}\cdot d\vec{l}=\int_{C_1}\vec{E}\cdot d\vec{l}+\int_{C_2}\vec{E}\cdot d\vec{l}=\int_{C_1}\vec{E}\cdot d\vec{l}=0 \tag{17}$$

in which C_1 is the line of force starting and ending at S_i and C_2 is a curve joining the above-mentioned points at S_i but with all its points lying outside S_i where $\phi=\phi_1$-ϕ_2 and the electric field vanishes. Let us notice that the electric field and the current have always the same direction and sense as vectors, because the electric conductivity is a positive scalar. In addition, as it is argued above, the current cannot reverse the sign of its component along the tangent vector of line of forces. Therefore, it follows that also the electric field cannot revert the sign of its component along a line of force. Thus, the integrand of the line integral over the C_1 curve should have a definite sign at all the points, hence implying that ϕ and the electric field should vanish exactly in all C_1. Resuming, it follows that the electric field vanish also at the interior of any of the closed surfaces S_i. Furthermore, as the sources within the closed surfaces are assumed to be bounded surface densities of charges, the vanishing of the electric field both at outside and inside regions implies that these densities are vanishing. Note that for double layer densities this is not true. As the above-mentioned sources are associated with $\phi=\phi_1$-ϕ_2, it follows that the evoked potential at B_0 uniquely fixes the assumed kind of sources generating it when they have their support in a set of non-nesting surfaces S.

IV. A 2D EXAMPLE

As it was announced in the Introduction, in order to illustrate the realization of the above discussion in a physical situation, in this section, a 2D electrostatic problem associated with a charge density distributed over a line (a point in the complex plane) parallel to a slotted and grounded planar conductor will be analyzed. For this purpose consider a localized density of filamentary charges given by

$$\rho_1(z)=\rho_1(x_1)\,\delta(x_2-h), \tag{18}$$

where $z=x_1+i\,x_2$ and $h>0$ is the height of the line supporting the charges over a slotted conductor plane sited at $x_2=0$. The slot is assumed to have width $2a$ and to be symmetrically centered at the origin $x_1=0$.

We will argue that if $\rho_1(x_1)$ is bounded and absolutely integrable, then the measuring of the potential $\phi_1(z)$ generated by the charge density (18) in a whole $x_2=h_1<0$ line situated below the conductor and parallel to it, uniquely determines the charge density ρ_1, whatever, but nonvanishing, the value of the slot width a is.

The defined electrostatic problem has an analytic solution that can be written by using the Schwartz-Christoffel transformation (see Ref. [10]) to be

$$\phi_1(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx_1'G(x_1,x_2;x_1',h)\rho_1(x_1'), \tag{19}$$

where the Green function of the problem G has the explicit expression

$$G(x_1,x_2;x_1',x_2')=\frac{1}{2\pi}\ln\left(\frac{x_1-x_1'+\sqrt{(x_1+i\,x_2)^2-a^2}-\sqrt{[(x_1'+i\,x_2')^2-a^2]^*}+i(x_2+x_2')}{x_1-x_1'+\sqrt{(x_1+i\,x_2)^2-a^2}-\sqrt{(x_1'+i\,x_2')^2-a^2}+i(x_2-x_2')}\right), \tag{20}$$

in which the square root is defined as

$$\sqrt{z}\equiv|z|\exp\left(\frac{\arg(z)}{2}\right),$$

$$0\leqslant\arg(z)<2\pi,$$

and $2a$ is the width of the slot.

The equipotential lines associated to a line of charges situated at the point $z=0.5+i$ are illustrated in Fig. 4 for the case $a=0.5$.

Let us show in what follows two properties that will imply the uniqueness of the density of the form (18) under a definite outcome for the measurement of the potential at any line $x_2=h_1<0$.

A. Property 1

If a density of filamentary charges $\gamma(x_1')$ lying in a line L is bounded and absolutely integrable, then the potential that it defines in free space in another line L' parallel to L cannot exactly vanish along L'.

In order to show this statement, let us consider L to be the x_1 axis. Then the potential at any point of the 2D plane will be

$$\phi_\gamma(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx_1'\,\gamma(x_1')\ln[x_2^2+(x_1-x_1')^2]^{1/2}, \tag{21}$$

and let us consider that $\phi_\gamma(z)$ is vanishing at some parallel plane $x_2=b$. Then, after taking a first derivative of $\phi_\gamma(z)$ over x_1 it follows that

$$0=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx_1'\,\gamma(x_1')\frac{(x_1-x_1')}{b^2+(x_1-x_1')^2}, \tag{22}$$

for all x_1. Therefore, after further Fourier transforming Eq. (13) over x_1 and employing

$$\int_{-\infty}^{\infty}dx\,\exp(ixq)\,\frac{x}{b^2+x^2}=i\pi\exp(-|q|b)(q), \tag{23}$$

produces

$$0=i\pi\exp(-|q|b)\operatorname{sgn}(q)\,\gamma(q), \tag{24}$$

where $\gamma(q)$ is the Fourier transform of the spatial density $\gamma(x_1)$. Therefore, it follows that $\gamma(q)$ is null for almost all the q values and hence the spatial density $\gamma(x_1)$ also vanishes. This completes the proof of Property 1.

The second property to be used is discussed below.

B. Property 2

If a density of filamentary charges $\gamma(x_1)$ defined on a line L, is bounded and absolutely integrable, then the normal component of the electric field that it defines in another line L' parallel to L cannot exactly vanish within an open interval of L'.

To start the proof, let us consider the potential expression in Eq. (21) as evaluated in a neighborhood of the same line L' defined by $x_2=b$. Taking a derivative over x_2 and evaluating at L' defines the normal component of the electric field on the line L'. This quantity takes the form

$$E_n(x_1,b)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx_1'\,\gamma(x_1')\frac{b}{b^2+(x_1-x_1')^2}. \tag{25}$$

Let us study the implications of assuming the vanishing of E_n in a whole open neighborhood N defined by $-a_1<x_1<a_1$. In the present case, as the Eq. (25) is not vanishing for all the line L', the recourse of Fourier transforming the expression (25) is not appropriate. However, it is possible to perform an arbitrary number of derivatives of Eq. (25) over the x_1 variable to have in N,

$$0=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx_1'\frac{d^n}{dx_1'^n}\,\gamma(x_1')\frac{b}{b^2+(x_1-x_1')^2}. \tag{26}$$

After substituting $\gamma(x_1')$ in terms of its Fourier transform through

$$\gamma(x)=\int_{-\infty}^{\infty}\frac{dq}{2\pi}\exp(-ixq)\,\gamma(q), \tag{27}$$

and using

$$\int_{-\infty}^{\infty}dx\,\exp(ixq)\,\frac{1}{b^2+x^2}=\frac{\pi}{b}\exp(-|q|b),$$

it follows for all the integers n

$$0=\int_{-\infty}^{\infty}dq\,q^n\exp(-|q|b)\,\gamma(q)\exp(-ix_1q),\quad x_1\in N. \tag{28}$$

Further, after multiplying by the function

$$f(x_1)=\theta(x_1+a_1)-\theta(x_1-a_1),$$

where θ is the Heaviside step function and by also taking into account

$$\int_{-\infty}^{\infty}dx\,f(x)\exp(ixq)=2\frac{\sin(a_1q)}{q},$$

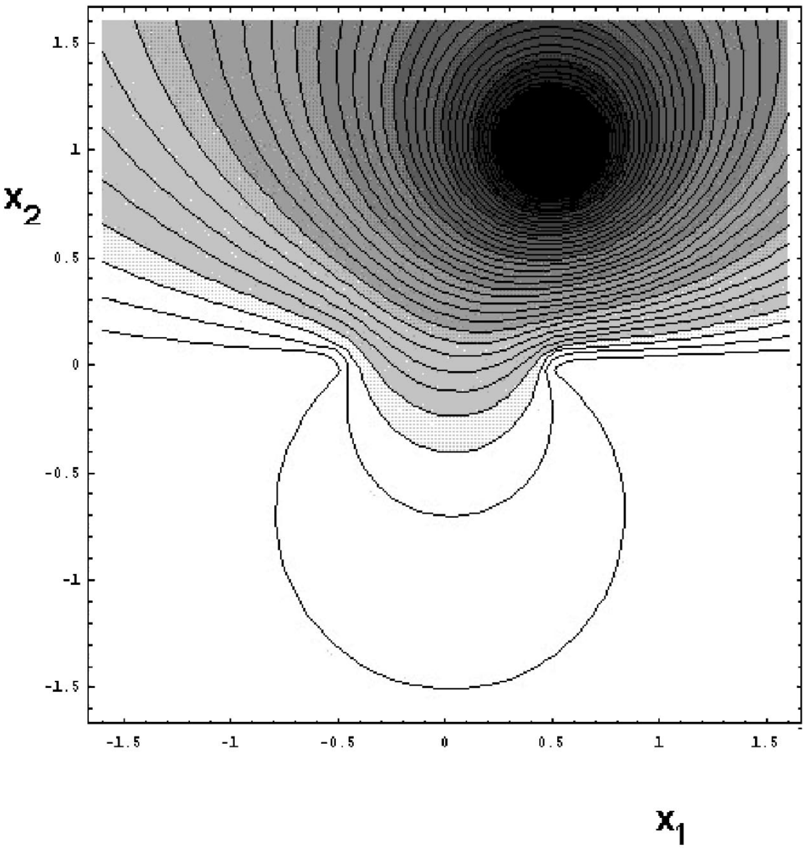

FIG. 4. The constant potential lines of a line of charge at position $z=1+i$ in the complex plane, situated over a conductor plane with a slot of width $2a=1$.

the following relation arises:

$$0=\int_{-\infty}^{\infty} dq\, q^n \exp(-|q|b)\,\frac{\sin(a_1 q)}{q}\,\gamma(q)$$

$$=\int_{-\infty}^{\infty} dq\, q^n g(q), \quad \text{for all } n>0. \tag{29}$$

In addition, the arbitrariness of n leads to

$$\int_{-\infty}^{\infty} dq \exp\left(-\frac{q^2}{2}\right) H_m(q) g(q)=0, \quad \text{for all } m\geqslant 0. \tag{30}$$

Expression (30) means that all the coefficients of the expansion of the functions g in series of the Hermite functions vanish. Thus, it follows that

$$\exp(-|q|b)\,\frac{\sin(a_1 q)}{q}\,\gamma(q)=0. \tag{31}$$

But this relation, in turns, implies that $\gamma(q)=0$ except perhaps at the zero measure set Z formed by the points $q_m=m\pi/a_1$, $-\infty<m<\infty$, $m\neq 0$, where $\sin(a_1q)/q=0$. After taking into account that the charge density is bounded and absolutely integrable, it follows that $\gamma(q)$ should be also bounded. This property, in turns implies that even when γ would not vanish at the points of Z, the Fourier inverse transform of $\gamma(q)$, that is the spatial density will vanish in the whole line L. This finishes the proof of the second property.

In what follows the above conclusions will be employed to show the uniqueness of the charge density (18) that determines specific potential values measured at all the points on another plane lying below the slotted conductor. This conclusion will be valid independently of the size of the slot a whenever it is finite. However, strictly at $a=0$ the argument becomes invalid and the lack of uniqueness is clear: independently of the value of the charge density (18), the perfect grounded conductor plane without any slot fully screen out the field from the zone below the real axis.

In order to discuss the uniqueness, two different charge densities ρ_1 and ρ_2 will be assumed to create the same measured potential within a plane $x_2=h_1<0$. Thus, the charge density

$$\rho(x_1)=\rho_1(x_1)-\rho_2(x_1) \tag{32}$$

(defined along the line $x_2=h$) will produce a vanishing potential at the points of the line $x_2=h_1$. Let us call $\phi(x_1,x_2)$ the potential created by the density ρ and assume that the size of the slot a is arbitrary but different form zero. Define also the potential $\phi_0(x_1,x_2)$ as the one corresponding to the same charge density ρ but when the conductor plane has no slot. After that, the solution ϕ related to the slotted plane can be equivalently expressed as

$$\phi(x_1,x_2)=\phi(x_1,x_2)-\phi_0(x_1,x_2)+\phi_0(x_1,x_2)$$

$$=\Phi(x_1,x_2)+\phi_0(x_1,x_2). \tag{33}$$

It is clear that $\phi_0(x_1,x_2)=0$ for any point below the real axis where $x_2<0$. As it was already noticed above, this is so because the perfect ground plane fully screen the potential created by the charge ρ. As for the potential Φ, it is created by charges that are localized precisely at the coordinate axis $x_2=0$. The charge density associated with this potential being equal to the difference of the densities generating the potentials ϕ and ϕ_0, has as its support the real axis $x_2=0$. This occurs because the charge density ρ lying over the plane $x_2=h$ is common to both solutions ϕ and ϕ_0 and then cancel in calculating the difference of the charges that generates the potential Φ.

It also follows that the whole field in the region below the conductor plane should coincide with Φ since the ϕ_0 is exactly vanishing there. Henceforth, as the sources of the field Φ are completely planar ones, they should vanish exactly due to Property 1 because by assumption the field difference ϕ of the two supposedly existing different charge densities ρ_1 and ρ_2, is vanishing in the measuring plane $x_2=h_1<0$.

Thus the first curious conclusion arises: the charge density concentrated in the slotted plane should exactly coincide with the perfect screening charge density of the conductor plane related with the potential ϕ_0. Moreover, as the charge density lying in the slotted plane is vanishing at the points within the slot, it follows that the charge density of the screening solution ϕ_0 at such points should also vanish. Furthermore, as the total field at the points being below the real axis is vanishing in such conditions, it also follows that by continuity the total electric field at the points of the slot in the $x_2=0$ line should also vanish identically.

It is useful to notice now that the normal component of the electric field at the point of the interior of the slot can only be generated by the charges above the real axis. In other words, the normal component should be produced by the charges lying at the plane $x_2=h$ which are associated with the difference between the two charge densities producing the same measured potential. This property follows because the charges contained in the real axis, as they were determined to vanish at the slot points, can only produce a tangential net field at the slot points lying along the real axis.

Thus it can be concluded that the planar charges, associated with the density $\rho(x_1)=\rho_1(x_1)-\rho_2(x_1)$ defined on the line $x_2=h$, create a vanishing normal component of the electric field at some open interval, say,

$$-a<-a_1<x_1<a_1<a$$

fully contained within the slot. However, Property 2 excludes this possibility, then implying that $\rho(x_1)=\rho_1(x_1)-\rho_2(x_1)=0$. Therefore, the uniqueness of the sources defining a measured potential values in a whole line lying below the real axis, follows however small would be the size of the slot a. This conclusion then illustrates the drastic change in uniqueness of the considered electrostatic problem produced by destroying, through an arbitrarily weak perturbation, the nested character of the charge distribution generating the fields.

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